MATH 2050C Lecture 7 (Feb 8)

Done: $\mathbb{R}$ is the complete ordered field.

GOAL: Define limit of a sequence $\lim (x_n) = x$ and study limit properties

Def^n: A sequence of real numbers is a function

$$X : \mathbb{N} \longrightarrow \mathbb{R}$$

Denote: $X(1) := x_1$, $X(2) =: x_2, \ldots, X(n) =: x_n$

Write: $X = (x_n) = (x_1, x_2, x_3, \ldots)$

CAUTION: Sets $\neq$ Sequences

E.g.) $((-1)^n) = (-1, 1, -1, 1, \ldots)$ *ordered & infinite*

$\{(-1)^n : n \in \mathbb{N}\} = \{-1, 1\}$ *unordered & maybe finite*

Examples of sequences

(1) constant seq. $(1, 1, 1, 1, 1, \ldots)$

(2) geometric seq. $(\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \ldots) = (\frac{1}{2^n})$

(3) arithmetic seq. $(1, 5, 9, 13, 17, \ldots) = (4n-3)$

$\rightarrow$ $^{+ve}$ odd seq. $(1, 3, 5, 7, 9, \ldots)$

$^{+ve}$ even seq. $(2, 4, 6, 8, 10, \ldots)$

(4) **Fibonacci seq.** ("inductively defined")

$$x_1 := 1 \quad ; \quad x_2 := 1$$

$$x_n := x_{n-1} + x_{n-2} \quad \text{for } n \geq 3$$

$$(x_n) = (1, 1, 2, 3, 5, 8, 13, 21, \ldots)$$
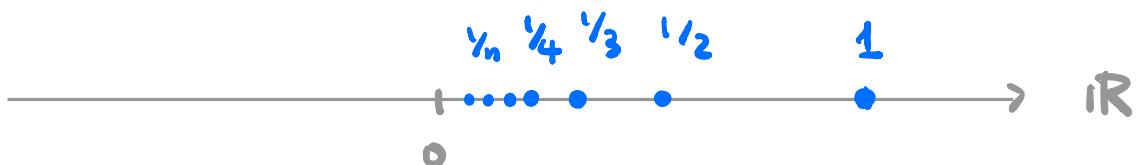
Q: How to define the limit of a sequence?

Simplest example:

$$(x_n) = \left(\frac{1}{n}\right) = \left(1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots, \frac{1}{100}, \ldots\right)$$

We want to say: "$\lim\left(\frac{1}{n}\right) = 0$"

because x_n "eventually" are getting "close" to 0

quantify this!

Def$\underline{^n}$: ($\varepsilon - K$ definition for limit)

We say that (x_n) **converges to** $x \in \underline{\mathbb{R}}$ (finite number)

Notation: $\lim (x_n) = x$ or $\lim_{n \to \infty} x_n = x$ or $x_n \to x$

iff $\forall \underset{(small)}{\varepsilon} > 0$, $\exists \underset{(large)}{K} = K(\varepsilon) \in \mathbb{N}$ s.t.

$$| x_n - x | < \varepsilon \qquad \forall n \geqslant K$$

(i.e. $x - \varepsilon < x_n < x + \varepsilon$)

Picture :
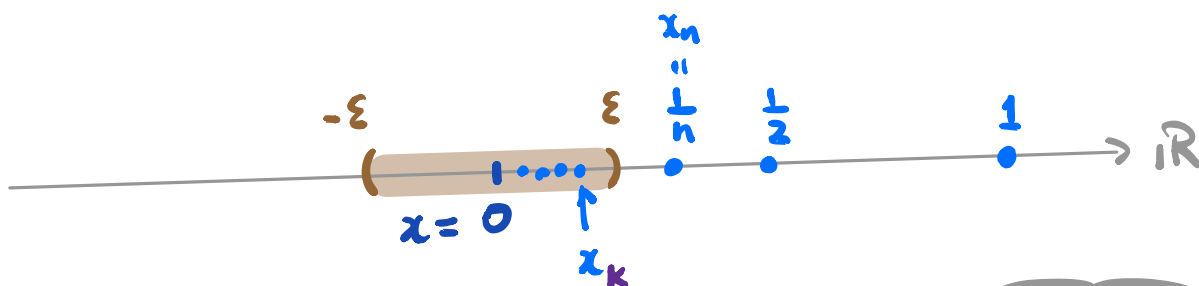
Example : $\boxed{\lim \left(\frac{1}{n} \right) = 0}$

$| \frac{1}{n} - 0 | < \varepsilon$
$\Longleftrightarrow \frac{1}{n} < \varepsilon$
$\Longleftrightarrow \frac{1}{\varepsilon} < n$

Let $\varepsilon > 0$ be fixed but arbitrary.

By **Archimedean Property**, we can choose

$K \in \mathbb{N}$ s.t. $K > \frac{1}{\varepsilon}$. Then, $\forall n \geqslant K$,

$$\left| \frac{1}{n} - 0 \right| = \frac{1}{n} \leqslant \frac{1}{K} < \varepsilon \qquad \underline{\qquad} \square$$

Example:

$$\lim \left(\frac{3n+2}{n+1}\right) = 3$$

Let $\varepsilon > 0$ be fixed but arbitrary.

Choose $K \in \mathbb{N}$ s.t. $K \overset{*}{>} \frac{1}{\varepsilon}$ (by Archimedean Property)

Then, $\forall n \geq K$,

$$\left| \frac{3n+2}{n+1} - 3 \right| = \left| \frac{(3n+2) - 3(n+1)}{n+1} \right| = \left| \frac{-1}{n+1} \right|$$

$$= \frac{1}{n+1} < \frac{1}{n} \leq \frac{1}{K} \overset{*}{<} \varepsilon$$

$\blacksquare$

Def^n: Given a seq (x_n) of real numbers, we say

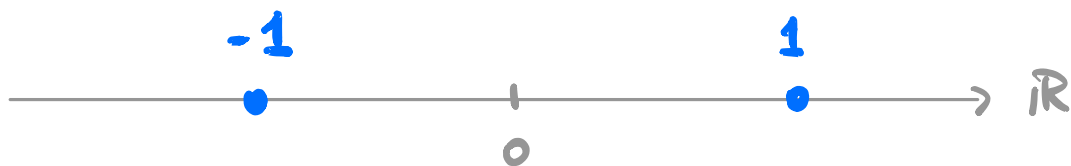(i) (x_n) is convergent if $\exists x \in \mathbb{R}$ s.t. $\lim (x_n) = x$

(ii) (x_n) is divergent if (x_n) is NOT convergent

i.e. $\nexists x \in \mathbb{R}$ s.t. $\lim (x_n) = x$

Example of divergent seq

Example: $((-1)^n)$ is divergent.

Picture:

$$-1 \qquad\qquad 1$$

(number line with points at -1 and 1, origin 0 marked, arrow to $\mathbb{R}$)

Why 1 is not the limit?

If $\lim((-1)^n) = 1$, then

$$\Big[\, \forall \varepsilon > 0, \ \exists K \in \mathbb{N} \ \text{st} \ |(-1)^n - 1| < \varepsilon \, \Big] \ \cdots \ (*)$$

($\forall n \geq K$ labeled above)

But for n odd, $|(-1)^n - 1| = |(-1) - 1| = 2$

So $(*)$ cannot be true.

Take $\varepsilon = 1 > 0$, if $(*)$ is true, $\exists K = K(1) \in \mathbb{N}$,

but if we choose $n \geq K$ odd, then

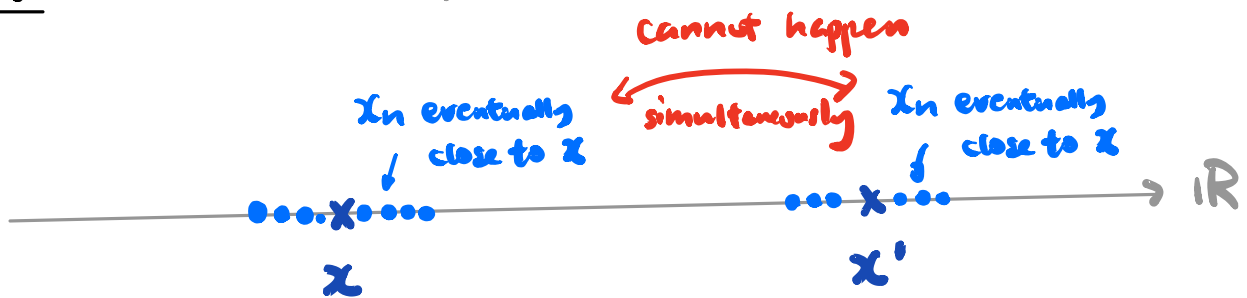$$|(-1)^n - 1| = 2 \neq 1 \qquad \text{Contradiction!}$$

Similarly, -1 is **NOT** the limit either. (Pf: Exercise)

Actually, $\nexists x \in \mathbb{R}$ st $\lim((-1)^n) = x$.

(Pf later).

Prop: Any convergent seq. (x_n) has a **unique** limit.

Proof: Idea: Suppose there are two limits x, x'

x_n eventually close to x — cannot happen simultaneously — x_n eventually close to x'

$\mathbb{R}$

x x'

Suppose **NOT**. then $\exists\ x \neq x' \in \mathbb{R}$ st.

$$\lim (x_n) = x \qquad \& \qquad \lim (x_n) = x'$$

Consider $\varepsilon := \dfrac{|x - x'|}{4} > 0$. by def^n of limit,

- $\lim (x_n) = x \Rightarrow \exists\ K = K(\varepsilon) \in \mathbb{N}$ st.

  (for the same ε)

  $$|x_n - x| < \varepsilon \qquad \forall n \geq K$$

- $\lim (x_n) = x' \Rightarrow \exists\ K' = K'(\varepsilon) \in \mathbb{N}$ st.

  $$|x_n - x'| < \varepsilon \qquad \forall n \geq K'$$

Take $\overline{K} := \max\{K, K'\} \in \mathbb{N}$. Then, $\forall n \geq \overline{K}$,

$$|x - x'| = |(x_n - x) - (x_n - x')| \qquad \text{Contradiction!}$$

(△-ineq)
$$\leq |x_n - x| + |x_n - x'| < \varepsilon + \varepsilon = \frac{|x - x'|}{2}$$

Now, we come back to

Claim: $((-1)^n)$ is divergent.

Pf: Suppose NOT, i.e. $((-1)^n)$ is convergent.

By Prop. $\exists$ unique limit $x = \lim((-1)^n)$.

By def^n of limit. $\forall \varepsilon > 0$, $\exists K \in \mathbb{N}$ s.t.

$$|(-1)^n - x| < \varepsilon \qquad \forall n \geq K.$$

Fix $\varepsilon = 1 > 0$, then $\exists K \in \mathbb{N}$ s.t.

$$|(-1)^n - x| < 1 \qquad \forall n \geq K.$$

Fix $n_1, n_2 \geq K$ s.t. n_1 odd, n_2 even. Then

$$2 = |-1-1| = |(-1)^{n_1} - (-1)^{n_2}|$$

$$= |((-1)^{n_1} - x) - ((-1)^{n_2} - x)|$$

$$\leq |(-1)^{n_1} - x| + |(-1)^{n_2} - x|$$

$$< 1 + 1 = 2$$

Contradiction!

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More examples

Example: Let $a > 0$. Then $\boxed{\lim\left(\frac{1}{1+na}\right) = 0}$

Let $\varepsilon > 0$ be fixed but arbitrary.

Choose $K \in \mathbb{N}$ s.t. $K > \frac{1}{a\varepsilon} > 0$

Then, $\forall n \geq K$.

$$\left|\frac{1}{1+na} - 0\right| = \left|\frac{1}{1+na}\right|$$

$$= \frac{1}{1+na} < \frac{1}{na} \leq \frac{1}{Ka} < \varepsilon$$

$$\left|\frac{1}{1+na} - 0\right| < \varepsilon$$
$$=$$
$$\frac{1}{1+na} < \frac{1}{na} \leq \frac{1}{Ka} < \varepsilon$$
$$\underline{\underline{}}$$
$$K > \frac{1}{a\varepsilon}$$

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Example: Let $b \in (0,1)$. Then $\boxed{\lim(b^n) = 0}$

Let $\varepsilon > 0$ be fixed but arbitrary.

Choose $K \in \mathbb{N}$ s.t. $K > \frac{\log \varepsilon}{\log b}$

Then, $\forall n \geq K$.

$$|b^n - 0| = b^n \leq b^K < \varepsilon$$

Want: $|b^n - 0| < \varepsilon$

$$|b^n - 0| = b^n \leq b^K < \varepsilon$$
$$\underline{\underline{\text{Solve for } k}}$$

$$K \log b < \log \varepsilon$$

$$K > \frac{\log \varepsilon}{\log b}$$

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